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Wave generation by an oscillating surface-pressure and its application in wave-energy extraction

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A two-dimensional analysis, based on linear surface-wave theory, is developed for an oscillating-water-column wave-energy device in water of arbitrary constant depth. The immersed part of the structure is assumed of shallow draught except for a submerged vertical reflecting wall. Both the cases of linear and nonlinear power take-off are considered. The results show that air compressibility can be important in practice, and its effects may in general be satisfactorily represented by linearization. The analysis indicates that using a turbine whose characteristic exhibits a phase difference between pressure and flow rate may be a method of strongly reducing the chamber length and turbine size with little change in the capability of energy extraction from regular waves. It was found in two examples of devices with strongly nonlinear power take-off that the maximum efficiency is only marginally inferior to what can be achieved in the linear case.

1. Introduction

A time-varying pressure applied on the water free-surface, in the presence of a train of progressing waves, will in general result in a non-zero exchange of energy. This constitutes the basis of several devices for sea-wave-energy utilization, namely the so-called oscillating-water-column devices, in which the reciprocating flow of air, displaced by the free surface within a chamber open at the immersed bottom, drives an air turbine (see e.g. Moody 1979). In an attempt to model the hydrodynamics of such devices, some authors neglected the spatial variation of the interior free surface, which was supposed to move as if acted on by a weightless piston. Examples of this approach are the works by Evans (1978) and Count *et al.* (1981), where the piston model is adopted on the grounds of the width of the inside free surface being small compared with the wavelength.

Lamb (1905) outlined, and Stoker (1957) worked out in detail, the two-dimensional theory of the waves generated in deep water by an oscillating pressure applied uniformly over a segment of the free surface, where the spatial variation of the surface is correctly accounted for. More general expressions were given by Wehausen & Laitone (1960), including the cases of two- and three-dimensional non-uniform oscillating pressure distributions on water of finite depth. The two-dimensional problem of an oscillating surface-pressure applied uniformly between two equally submerged vertical plates was dealt with by Ogilvie (1969).

Evans (1982) considered the general case of wave-power absorption by two- and three-dimensional systems of oscillating surface-pressure distributions, including the diffraction due to submerged structures. He derived reciprocity relations for the applied-pressure and diffraction properties, and presented general expressions for the time-averaged power developed by the pressure forces, which were then applied to special cases of single internal free surfaces of simple shapes when the immersed part of the structure is of negligible draught. The more complicated problem of an oscillating body with chambers that enclose portions of the free surface was dealt with recently by Fernandes (1983), who derived reciprocity relations to connect the several modes of wave diffraction and radiation. He went on to obtain numerical results for an axisymmetric buoy with pneumatic wave-power absorption with the help of a singularity distribution method.

The purpose of the present paper is to analyse the combined effects of several factors believed to be relevant to the engineering design of oscillating-water-column devices, namely finite water depth, air compressibility, and a turbine characteristic that exhibits a phase difference between pressure and flow rate or is nonlinear. In order to enable sufficiently simple analytical expressions to be derived, a two-dimensional geometry is adopted, and the wave diffraction due to the immersed part of the structure is ignored, except for the reflection by a wall behind the chamber extending vertically from the free surface to the bottom.

In §2 expressions are obtained for the waves generated by an oscillating uniform surface-pressure, which can be considered as extending to water of finite depth those given by Stoker (1957). In §3 the previous results are applied to derive expressions for the power absorption from a train of regular incident waves, including the case when a vertical reflecting wall is present behind the chamber. The air-flow rate in the chamber, due to the motion of the water surface, produces useful work by means of an air turbine or equivalent device, whose instantaneous mass-flow rate is assumed to be a known function of the pressure difference. The analysis is further developed and numerical results are presented, in §4, for the linear case when the springlike effect of air compressibility is assumed of constant stiffness, and the mass-flow rate of air through the turbine is taken proportional to the pressure difference. The proportionality constant is allowed to be complex, in order to model a turbine with non-zero phase difference between flow rate and pressure. The problem of nonlinear power take-off is considered in §5.

2. Wave generation by an oscillating surface-pressure

This section is concerned with the two-dimensional analysis of the waves generated by a simple time-harmonic pressure applied uniformly over a free-surface segment. The depth h is supposed to be constant. We assumed irrotational motion, with the usual linearized boundary conditions at the free surface, and chose a system of coordinates (x, y) with the positive y-direction pointing vertically upwards and the origin at the undisturbed free surface. The surface-pressure distribution is given by

$$p(x,t) = X(x)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where $X(x) = 0$ for $|x| > a$, and $X(x) = 1$ for $|x| < a$. Here, and whenever a physical quantity is equated to a complex expression, only the real part is to be taken, in accordance with the usual convention. In this section no submerged barriers are assumed to be present.

The velocity potential can be written as $\Phi(x, y, t) = \phi(x, y)\,\mathrm{e}^{\mathrm{i}\omega t}$, where ϕ is a complex function satisfying Laplace's equation $\phi_{xx}+\phi_{yy} = 0$. The linearized boundary conditions at the free surface are

$$\frac{\partial \Phi(x,0,t)}{\partial t}+\frac{p}{\rho}+g\eta = 0, \quad \Phi_y(x,0,t) = \frac{\partial \eta}{\partial t}, \tag{2}$$

where $\eta(x,t)$ is the free-surface elevation, ρ is the water density, and g the acceleration of gravity. Additional conditions are $\phi_y(x,-h)=0$, at the bottom, and the radiation condition (only outgoing waves are generated). A solution to a similar, more general problem, in which an arbitrary function of x instead of $X(x)$ is assumed for the pressure distribution, is given by Wehausen & Laitone (1960, p. 597). From it we obtain, for our case, $\eta=\eta_1+\eta_2$, where

$$\eta_1(x,t)=\frac{2\mathrm{e}^{\mathrm{i}\omega t}}{\pi\rho g}\int_0^\infty\frac{\cos kx\sin ka}{\alpha\coth kh-k}\,\mathrm{d}k, \tag{3}$$

$$\eta_2(x,t)=\frac{2\mathrm{i}m}{\rho g}\sin Ka\,\mathrm{e}^{\mathrm{i}\omega t}\cos Kx. \tag{4}$$

In the expressions above $\alpha=\omega^2/g$, and K is the real solution of $K\tanh Kh=\alpha$, which can easily be recognized as the wavenumber for a regular train of waves of angular frequency ω. The constant $m=(1+\alpha h\,\mathrm{cosech}^2\,Kh)^{-1}$. The integral of (3) can conveniently be transformed with the help of the function

$$f(k)=\frac{m^{-1}(k-K)}{k-\alpha\coth kh}, \tag{5}$$

which is such that $f(K)=1$. For finite values of h it can be found that $f(k)$ is continuous, and its first derivative is finite, in the interval $-\infty<k<\infty$. Integrating by parts in (3), we find

$$\eta_1(x,t)=\frac{m\,\mathrm{e}^{\mathrm{i}\omega t}}{\pi\rho g}[F(a+x)+F(a-x)], \tag{6}$$

where

$$F(x)=\sin Kx\int_0^\infty[f'(K+k)+H(K-k)f'(K-k)]\,\mathrm{Ci}\,(|x|\,k)\,\mathrm{d}k$$
$$-\mathrm{sgn}\,(x)\cos Kx\left\{\pi-\int_0^\infty[f'(K+k)-H(K-k)f'(K-k)]\,\mathrm{si}\,(|x|\,k)\,\mathrm{d}k\right\}. \tag{7}$$

In this equation $H(x)$ is Heaviside's unit step function (equal to unity for $x>0$ and to zero for $x<0$), $\mathrm{sgn}\,(x)=\pm1$ for $x\gtrless0$, and Ci ( ) and si ( ) = Si ( ) $-\frac{1}{2}\pi$ are the cosine integral and sine integral respectively. In the first integral of (7) the function Ci has only a logarithmic singularity (for zero argument) which is integrable. When $k\rightarrow\infty$ it is $f'(k)=O(k^{-2})$, and this, together with the well-known asymptotic behaviour of the functions Ci and si, ensures a quick convergence of the integrals.

In (7) the functions Ci and si vanish for $|x|\rightarrow\infty$, and the same is true for the two integrals. The following far-field expression is then easily obtained for the surface elevation:

$$\eta(x\rightarrow\pm\infty,t)=\frac{2\mathrm{i}m}{\rho g}\sin Ka\,\mathrm{e}^{\mathrm{i}(\omega t-K|x|)}, \tag{8}$$

which can be seen to represent outgoing waves.

In order to obtain the expression for the power extracted from the waves and the corresponding efficiency, we must know the air-flow rate displaced by the water free-surface inside the chamber, i.e. we have to calculate the spatially averaged value $\bar{\eta}(t)$ of the free-surface elevation in the interval $-a<x<a$. The part of $\bar{\eta}$ corresponding to η_1 can be obtained from (3) by performing first the integration with respect to x and then following a procedure similar to that used to derive (6) from (3). We find

$$\bar{\eta}(t)=\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\rho gKa}(u+2\mathrm{i}m\sin^2 Ka), \tag{9}$$

where
$$u=\frac{2K}{\pi}\int_0^\infty \frac{\sin^2 ka}{k(\alpha\coth kh-k)}\,\mathrm{d}k.$$

By integrating by parts, the latter equation can be written in the following form, which is more convenient numerically:

$$\begin{aligned}u=\frac{m}{\pi}\Big\{\int_0^\infty f'(k)\Big[\ln\frac{|k-K|}{k}+\mathrm{Ci}\,(2ka)\Big]\,\mathrm{d}k\\ -\cos(2Ka)\int_0^\infty [f'(K+k)+H(K-k)f'(K-k)]\,\mathrm{Ci}\,(2ka)\,\mathrm{d}k\\ -\sin(2Ka)\Big[\pi-\int_0^\infty [f'(K+k)-H(K-k)f'(K-k)]\,\mathrm{si}\,(2ka)\,\mathrm{d}k\Big]\Big\}.\end{aligned}\tag{10}$$

In the case of deep water ($Kh\rightarrow\infty$), we find the limiting values $K=\alpha$, $m=1$, and, instead of (7) and (10), we obtain the simpler expressions

$$F(x)=\sin\alpha x\,\mathrm{Ci}\,(\alpha|x|)-\mathrm{sgn}\,(x)\cos\alpha x\,[\pi+\mathrm{si}\,(\alpha|x|)],\tag{11}$$

$$u=-\frac{1}{\pi}\{\cos(2\alpha a)\,\mathrm{Ci}\,(2\alpha a)+\sin(2\alpha a)\,[\pi+\mathrm{si}\,(2\alpha a)]-\ln(2\alpha a)-\gamma\},\tag{12}$$

where $\gamma=0.5772\ldots$ is Euler's constant. In the shallow-water approximation ($Kh\rightarrow 0$) it is $m=\frac{1}{2}$, and $F(x)=-\pi\cos Kx$, $u=-\frac{1}{2}\sin(2Ka)$.

The coefficients A and B, as defined by Evans (1982), can be related to the above quantities by
$$B+\mathrm{i}\omega A=-2\mathrm{i}a\omega\bar{\eta}\,\mathrm{e}^{-\mathrm{i}\omega t}.$$

3. Wave-energy extraction

3.1. *Sinusoidal wave incident upon the chamber*

Now we consider the case when there is an incident train of regular waves, travelling from $x=-\infty$, defined by its elevation

$$\eta_i(x,t)=A\,\mathrm{e}^{\mathrm{i}(\omega t-Kx)},\tag{13}$$

which produces an oscillating air pressure $P(t)$ inside the chamber. The pressure difference $P(t)-P_a$ ($P_a\equiv$ outside pressure) is used to drive an air turbine (or equivalent device), whose mass-flow rate per unit length of wave crest $M(t)$ is supposed to be a known function of the pressure difference $M(t)=\Psi[P(t)-P_a]$.

Let $V(t)$ be the volume of air in the chamber, per unit length of wave crest, and assume that the chamber walls touch the free surface at $x=\pm a$ and are vertical (or at least parallel) near the free surface. Then we may write $V(t)=2a[H-z(t)]$, where $z(t)$ is the free-surface elevation averaged over the segment $-a<x<a$, and H is the chamber volume above the unperturbed free surface divided by the inside free-surface area. (H equals the level of the chamber ceiling measured from the unperturbed free surface if the chamber width is constant.) Denoting the air density inside the chamber by $\rho_c(t)$, we have

$$\begin{aligned}-\Psi[P(t)-P_a]&=\frac{\mathrm{d}}{\mathrm{d}t}[\rho_c(t)\,V(t)]\\&=2a\left\{\frac{\mathrm{d}\rho_c(t)}{\mathrm{d}t}[H-z(t)]-\rho_c(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right\}.\end{aligned}\tag{14}$$

In addition, the air density ρ_c is supposed to be a known function of the pressure P. A convenient approximation consists in assuming that the air flow is isentropic in the chamber, turbine and connecting ducts, in which case it is $\rho_c = \rho_a(P/P_a)^{1/\gamma}$, where ρ_a is the outside air density and $\gamma = c_p/c_v$ is the specific-heat ratio.

Equation (14) shows that, in general, the overall problem is nonlinear, owing to the varying air density ρ_c and to the nonlinearity of the turbine function Ψ. The pressure $P(t)$ is a periodic (in general non-simple-harmonic) function of time, which can be expanded in a Fourier series

$$P(t) = \sum_{n=0}^{\infty} P_n \,\mathrm{e}^{\mathrm{i}n\omega t}, \tag{15}$$

where the coefficients P_n are complex quantities to be determined from the geometrical and physical characteristics of the system. Comparing (15) and (1), and taking into account the linearity of the governing equations for the surface waves, we obtain the following expressions for the elevation of the resulting wave motion:

$$\eta_i(x,t) + \frac{P_a - P_0}{\rho g} + \sum_{n=1}^{\infty} P_n \,\eta_n(x,t),$$

and for the corresponding average value

$$z(t) = \frac{A}{Ka}\sin Ka \,\mathrm{e}^{\mathrm{i}\omega t} + \frac{P_a - P_0}{\rho g} + \sum_{n=1}^{\infty} P_n \,\bar{\eta}_n(t). \tag{16}$$

The subscript n in η_n and $\bar{\eta}_n$ means that these quantities can be obtained from the expressions given in §2 for η and $\bar{\eta}$ respectively if ω is replaced by $n\omega$. (More generally, the other quantities depending on ω, such as α, K, m, should be written as α_n, K_n, m_n, if $n > 1$.) The coefficients P_n can be calculated, at least in theory, from (14) and (16), together with the isentropic relation for ρ_c.

It is particularly important to know the efficiency of the energy absorption. First we note that, far away from the chamber ($|x| \to \infty$), the water motion can be considered as the superposition of three types of progressive waves. The time-averaged energy flux of the incident wave, per unit length of crest, is

$$E_i = \frac{|A|^2 \rho g \omega}{4mK}. \tag{17}$$

Besides the incident wave, we have the transmitted wave (subscript t) and the reflected wave (subscript r), whose surface elevations η_t and η_r, for $x \to \pm\infty$, can easily be found to be given by

$$\begin{bmatrix} \eta_t(x \to +\infty, t) \\ \eta_r(x \to -\infty, t) \end{bmatrix} = \begin{bmatrix} A + P_1 \dfrac{2\mathrm{i}m}{\rho g}\sin Ka \\ P_1 \dfrac{2\mathrm{i}m}{\rho g}\sin Ka \end{bmatrix} \mathrm{e}^{\mathrm{i}(\omega t - K|x|)} + \frac{2\mathrm{i}}{\rho g}\sum_{n=2}^{\infty} P_n m_n \sin K_n a \,\mathrm{e}^{\mathrm{i}(n\omega t - K_n|x|)}. \tag{18}$$

The corresponding expressions for the energy flux are

$$\begin{bmatrix} E_t \\ E_r \end{bmatrix} = \begin{bmatrix} \left|A + P_1 \dfrac{2\mathrm{i}m}{\rho g}\sin Ka\right|^2 \\ \left|P_1 \dfrac{2\mathrm{i}m}{\rho g}\sin Ka\right|^2 \end{bmatrix} = \frac{\rho g \omega}{4mK} + \frac{\omega}{\rho g}\sum_{n=2}^{\infty} |P_n|^2 \frac{n m_n}{K_n}\sin^2 K_n a. \tag{19}$$

The difference $E_i - (E_t + E_r)$ is the average power extracted from the waves. The efficiency is then $\epsilon = 1 - (E_t + E_r)/E_i$.

It can easily be found that the maximum value of ϵ is $\frac{1}{2}$ (in agreement with Evans

1982), for $P_1/\rho g A = (-4\mathrm{i}m \sin Ka)^{-1}$, $P_2 = P_3 = \ldots = 0$, the latter conditions implying a linear behaviour of the system. Higher efficiencies, however, can be expected by introducing a reflecting vertical wall, which will prevent energy being lost as transmitted waves. If $Ka = n\pi$ (n integer), then $\epsilon \leqslant 0$, this meaning that no net energy extraction is possible. (In this case $\epsilon = 0$ only if $P_2 = P_3 = \ldots = 0$; the value of P_1 does not affect ϵ, since pressure oscillations of frequency ω do not radiate energy in waves.)

3.2. *Reflecting vertical wall behind the chamber*

We consider now the oscillating surface-pressure to be applied over the interval $-(a+b) < x < -b$ $(b \geqslant 0)$, and assume a wall submerged vertically from the surface to the bottom, at $x = 0$. The wave field is restricted to the half-plane $x < 0$.

Following the same procedure as before, we consider first the radiated wave due to an oscillating pressure of unit amplitude. Its surface elevation can easily be found to be given by $\eta(x,t;a+b)-\eta(x,t;b)$, where $\eta(x,t;a)$ stands for the expressions derived in §2 for the case of an oscillating surface-pressure over the interval $-a < x < a$. The average, taken over the interval $-(a+b) < x < -b$, is

$$\bar{\eta}(t) = \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\rho g K a}(r + 2\mathrm{i}mq^2), \tag{20}$$

where

$$r = u(a+b)+u(b)+2u(\tfrac{1}{2}a)-2u(\tfrac{1}{2}a+b), \quad q = \sin K(a+b) - \sin Kb, \tag{21}$$

the expression for the function $u(a)$ being given by the right-hand side of (10) or (12).

The average elevation for the resulting wave motion, which includes the contribution from the standing wave due to the reflection of the incident wave on the wall, is then

$$z(t) = \frac{2Aq\,\mathrm{e}^{\mathrm{i}\omega t}}{Ka} + \frac{P_a - P_0}{\rho g} + \sum_{n=1}^{\infty} P_n\,\bar{\eta}_n(t). \tag{22}$$

Here the higher harmonics $\bar{\eta}_n$ are obtained from $\bar{\eta}$ (20) in the same manner as indicated in §3.1.

Equation (14) can be applied as in §3.1, provided that $2a$ is replaced by a, since the chamber length is now equal to a. Expressions for $\eta_r(x \rightarrow -\infty, t)$ and E_r can be derived as outlined in §3.1. The power available to the turbine is simply equal to $E_i - E_r$, and the efficiency is given by $\epsilon = 1 - E_r/E_i$.

It is not difficult to show that, if $q = 0$, i.e. if $Ka/2\pi$ is an integer or $K(a+2b)/\pi$ is an odd integer, then $\epsilon \leqslant 0$ ($\epsilon = 0$ only if $P_2 = P_3 = \ldots = 0$).

In the special case when $b = 0$ (chamber adjacent to a reflecting wall), it can be found that the far-field expression of the elevation for the reflected wave is identical with the expression (18) for η_t, it being understood that in this case it is taken for $x \rightarrow -\infty$. Besides, the expression for the average reflected energy flux E_r is identical with the expression (19) for E_t.

4. Linear problem

4.1. *Analytical expressions*

Simple analytical results can be obtained if the air-density variation is considered small and is linearized in terms of the pressure, which for isentropic flow leads to $\rho_c = \rho_a + (P - P_a)\rho_a/\gamma P_a$, and if, in addition, the air-turbine mass-flow rate M is assumed to be proportional to the pressure difference (linear turbine). In order to explore the eventual benefits of using a phase-controlled turbine, we consider, in general, a complex proportionality constant $C\,e^{i\theta}$ (C real positive, $|\theta| < \frac{1}{2}\pi$), defined by

$$C\,e^{i\theta} = \frac{\rho}{\rho_a}\left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{M}{P-P_a}, \tag{23}$$

where the real part of M is taken positive for outward flow. (For incompressible air, the coefficient λ defined by Evans (1982) is related to the quantities above by $\lambda = C\,e^{i\theta}\rho^{-1}(a/g)^{\frac{1}{2}}$.) Assuming z to be small compared with H, the governing equations can be linearized, and in particular (14) may be written as

$$\frac{C\,e^{i\theta}}{2\rho(ga)^{\frac{1}{2}}}(P-P_a) = -\frac{H}{\gamma P_a}\frac{dP}{dt} + \frac{dz}{dt}. \tag{24}$$

For the case considered in §3.1 (no reflecting wall), if account is taken of (9), (15) and (16), we easily find $P_0 = P_a$, $P_n = 0$ ($n = 2, 3, 4, \dots$) and

$$P_1 = -\frac{A\rho g \sin Ka}{u-\xi+w+iv}, \tag{25}$$

where u is given by (10) or (12), and

$$v = \frac{K}{2\omega}(ga)^{\frac{1}{2}} C\cos\theta + 2m\sin^2 Ka, \quad w = -\frac{K}{2\omega}(ga)^{\frac{1}{2}} C\sin\theta, \tag{26}$$

$$\xi = \frac{\rho g H K a}{\gamma P_a}. \tag{27}$$

The springlike effect of air compressibility, represented by ξ, is seen to be equivalent to adding an imaginary term to the turbine proportionality constant. The expression for the efficiency now becomes

$$\epsilon = 2(V - V^2 - U^2), \tag{28}$$

where

$$U = \frac{2m(u-\xi+w)\sin^2 Ka}{(u-\xi+w)^2+v^2}, \quad V = \frac{2mv\sin^2 Ka}{(u-\xi+w)^2+v^2}. \tag{29}$$

It can easily be seen that the efficiency ϵ reaches its maximum value, equal to $\frac{1}{2}$, when $U = 0$, $V = \frac{1}{2}$. For given values of $Ka \neq n\pi$ and Kh, these resonance conditions yield the following expressions for the optimum value of the turbine coefficient:

$$\tan\theta = \frac{u-\xi}{2m\sin^2 Ka}, \quad C = 2\left\{\frac{\tanh Kh}{Ka}[(u-\xi)^2 + 4m^2\sin^4 Ka]\right\}^{\frac{1}{2}}. \tag{30a,b}$$

If the air is assumed incompressible ($\xi = 0$), (30a,b) agree with the general condition for maximum power, derived by Evans (1982, equation (2.18)), which, for a single pressure-surface, can be written, in his notation, $\lambda = B - i\omega A$.

In the case when a reflecting wall is present, considered in §3.2, we find, instead of (25),

$$P_1 = -\frac{2A\rho g q}{r-\xi+2w+is},$$

where q and r are given by (21), and

$$s = \frac{K}{\omega}(ga)^{\frac{1}{2}} C\cos\theta + 2mq^2. \tag{31}$$

The efficiency is now $\epsilon = 1 - E_r/E_i$, and its expression is found to be

$$\epsilon = 4(S - S^2 - R^2), \tag{32}$$

where $$R = \frac{2m(r-\xi+2w)\,q^2}{(r-\xi+2w)^2+s^2}, \quad S = \frac{2msq^2}{(r-\xi+2w)^2+s^2}. \tag{33}$$

The maximum efficiency is $\epsilon = 1$, for $R = 0$, $S = \frac{1}{2}$, and the resonance value of the turbine coefficient is given by

$$\tan\theta = \frac{r-\xi}{2mq^2}, \quad C = \left\{\frac{\tanh Kh}{Ka}\,[(r-\xi)^2+4m^2q^4]\right\}^{\frac{1}{2}}. \tag{34a,b}$$

Equations (30*b*) or (34*b*) give the values of C that maximize ϵ, for any given values of Ka, Kh, Kb, ξ and θ; it can be found that this is true even if (30*a*) or (34*a*) are not satisfied.

4.2. *Numerical results*

4.2.1. *Chamber with adjacent wall*

We consider first the special case of a chamber of length a, adjacent to a reflecting wall ($b = 0$). It can easily be found that the values of the pressure amplitude P_1 and the efficiency ϵ are exactly twice the corresponding ones for a chamber of length $2a$ without reflector, the latter having the same values of Ka, Kh, ξ and θ as the former, and a turbine constant C two times larger. This means that (24)–(29) (and also (30*a*,*b*) defining the resonance conditions) can be applied, provided that P_1, ϵ and C are replaced by $\frac{1}{2}P_1$, $\frac{1}{2}\epsilon$ and $2C$ respectively. In particular, instead of (30*b*), we have

$$C = \left\{\frac{\tanh Kh}{Ka}\,[(u-\xi)^2+4m^2\sin^4 Ka]\right\}^{\frac{1}{2}}. \tag{35}$$

If no phase difference is allowed ($\theta = 0$), the pressure difference is simply proportional to the air mass-flow rate, a condition that is approximately fulfilled by the so-called Wells turbine (Gato & Falcão 1984). The resonance value of Ka (for $\epsilon = 1$) is then determined by $u-\xi = 0$. Figure 1 shows a plot of u as a function of a/λ ($\lambda = 2\pi/K \equiv$ wavelength) for several values of h/λ, as well as the straight lines representing ξ versus a/λ for different values of H (assuming as standard values $\rho = 10^3$ kg m^{-3}, $g = 9.8$ m s^{-2}, $\gamma = 1.4$ and $P_a = 10^5$ Pa). It can be seen that the number of positive roots of $u-\xi$ depends on h/λ and H. For constant h/λ the first positive root (i.e. the lowest resonance value of a for given λ) increases with H, until H reaches a critical value depending on h/λ, above which no positive roots exist, this implying that the efficiency ϵ cannot then attain unity. The critical value of H increases with h/λ and is approximately equal to 3.6, 5.7 and 10.5 m for $h/\lambda = 0.1$, 0.25 and ∞ respectively. These results show that, for fixed H, the air compressibility affects the efficiency more strongly for the smaller relative depths h/λ. In the case of incompressible air, the resonance condition reduces to $u = 0$. For deep water ($h/\lambda = \infty$), u has three positive roots, which are shown in table 1, together with the corresponding values of C given by (35). For a given wavelength, the first root is obviously the one that yields the smallest chamber width. However, it is interesting to remark that the second root allows a considerably smaller turbine diameter, since the turbine cross-sectional area is known to be roughly proportional to the volume-flow rate divided by the pressure difference, i.e. to $Ca^{\frac{1}{2}}$. Figure 2, valid for deep water and incompressible air, shows how the efficiency is affected by varying a/λ and C. Maximum efficiency, equal to unity, occurs for $a/\lambda = 0.2066$ and $C = 1.628 = C_1$ (cf. table 1). For $C = C_1$ the efficiency remains above 0.6 if $0.16 < a/\lambda < 0.38$. It can be seen that an oversized turbine ($C = 2C_1$) allows no higher efficiency for any value of a/λ, whereas a smaller turbine ($C < C_1$) can give better efficiencies for the larger

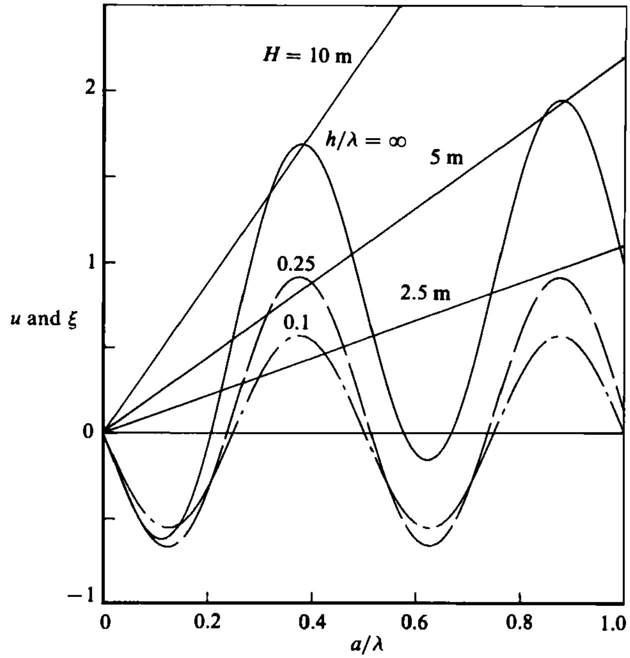

FIGURE 1. Variation of u (curved lines for several values of h/λ) and of ξ (straight lines for several values of H) with dimensionless chamber length a/λ.

a/λ	0.2066	0.5766	0.6669
C	1.628	0.2244	0.7340

TABLE 1

wavelengths. In what follows, whenever resonance working conditions are mentioned, it shall be understood that they refer to the lowest positive value of a/λ satisfying (30a) (or (34a) in §4.2.2). It is of interest to know the influence of the depth h, given the frequency rather than the wavelength. In figure 3, valid for incompressible air ($\xi = 0$), the solid line shows $\alpha a/2\pi$ as a function of αh, where $\alpha = \omega^2/g$ as before, and a satisfies the resonance condition $u = 0$. As αh decreases from infinity, it can be seen that αa increases by about 7 % to a maximum at $\alpha h/2\pi \approx 0.31$ ($h/\lambda \approx 0.32$) and then decreases to zero. It can also be seen that the ratio a/λ increases monotonically to 0.25, and the resonance value of the turbine constant, given by (35), decreases to zero, as αh decreases from infinity. These results show the advantage, in terms of reduction of chamber length and turbine size, of locating the energy-extracting device in shallow waters.

The discrepancy between results from the present (uniformly applied pressure) theory and the rigid-plate model (McCamy 1961) is illustrated in figure 4, for incompressible air and deep water. The curves show the efficiency ϵ as a function of a/λ, for a turbine constant $C = 1.628$. The difference is small for a/λ up to about the resonance value, but the rigid-plate model is clearly inadequate for larger values of a/λ, which agrees qualitatively with the corresponding results given by Evans (1982) for a circular disk.

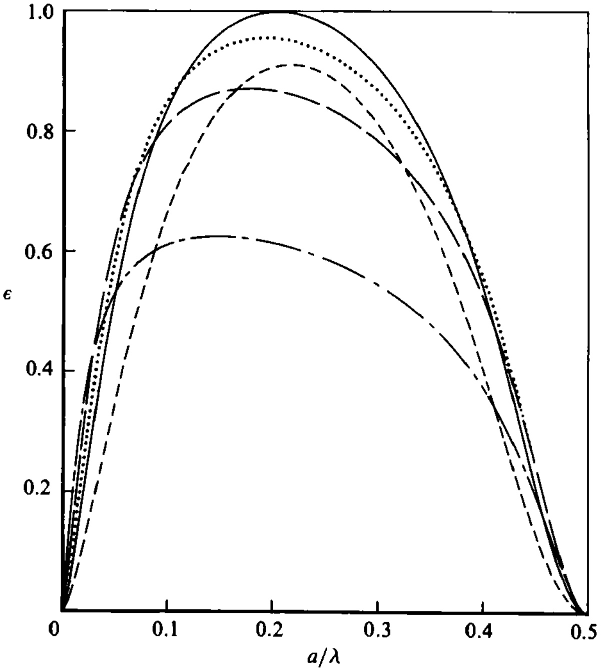

FIGURE 2. The efficiency ϵ for incompressible air and deep water, plotted against dimensionless chamber length a/λ, for $\theta = 0$ and different values of the turbine coefficient C: ——, $C = 1.628 = C_1$; ..., $0.7C_1$; — —, $0.5C_1$; —·—, $0.25C_1$; - - - -, $2C_1$.

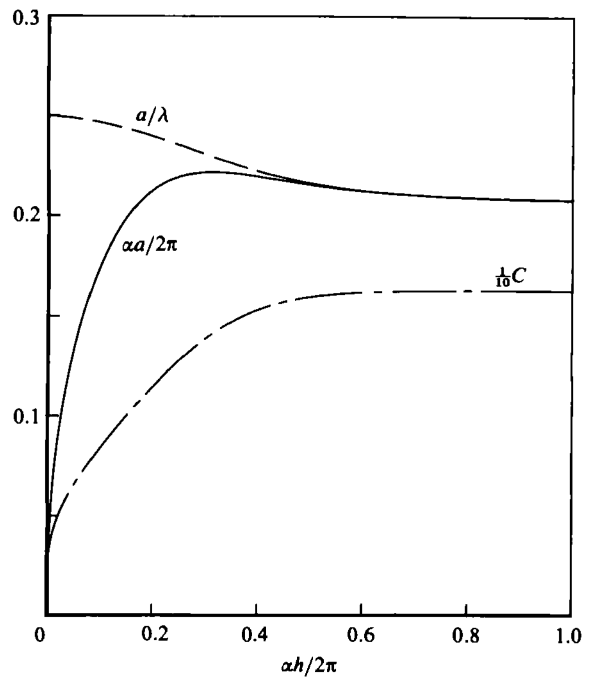

FIGURE 3. Turbine coefficient C and dimensionless values of chamber length a/λ and $\alpha a/2\pi$ shown as functions of dimensionless depth $\alpha h/2\pi$, for resonance conditions ((35) and $u = 0$), assuming incompressible air and $\theta = 0$.

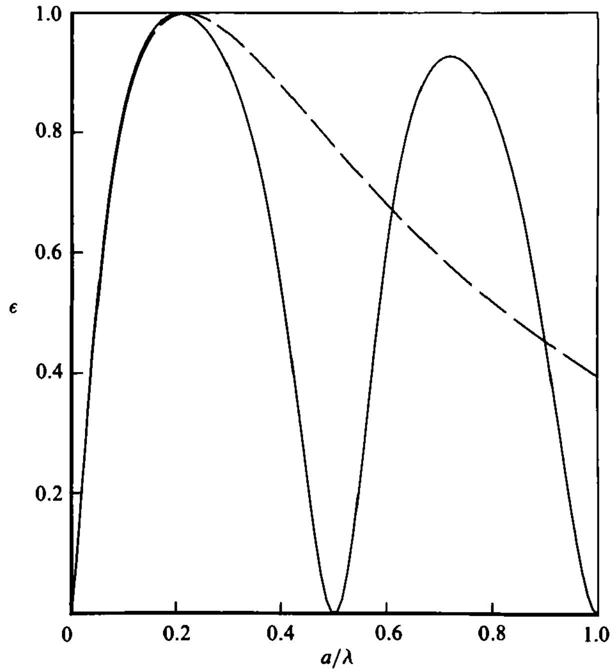

FIGURE 4. Variation of efficiency ϵ with dimensionless chamber length a/λ, in the cases of an oscillating uniformly applied pressure (solid line) and an oscillating rigid plate (dashed line), for deep water, incompressible air and turbine constant $C = 1.628$, $\theta = 0$.

If the phase difference θ is unrestricted, then efficiency equal to unity can be attained for any value of a/λ ($2a/\lambda \neq 1, 2, \ldots$), provided that θ and C satisfy (30*a*) and (35) respectively. A way that has been proposed for achieving $\theta \neq 0$ consists in employing an air turbine of the Wells type, with controllable rotor-blade stagger angle (Gato & Falcão 1983). Figures 5 and 6, valid for incompressible air, show the resonance values of θ and C respectively as functions of a/λ, for several values of h/λ. A non-zero phase difference implies that, during a fraction (equal to $|\theta|/\pi$) of the period, the air pressure increases across the turbine, which in fact then works as a compressor supplying energy to the wave field. The ratio τ between the work of compression and the work of expansion done by the turbine during one cycle can easily be found to be given by $\tau = (\pi|\tan\theta - \theta|^{-1} + 1)^{-1}$ (exactly for incompressible air, approximately otherwise). We note that, although efficiency equal to unity can be attained in theory regardless of the values of a/λ and hence of τ, in practice the time-averaged net power produced by the turbine will be severely reduced by dissipative effects occurring in the compression and expansion of air in the turbine, if τ is not restricted to small values. Taking, for example, $\tau = 0.15$ as the acceptable upper limit, we obtain $\theta = -0.999$ rad ($-57.2°$). Then, from figures 5 and 6, for incompressible air, we find the resonance values of a/λ and C, which are shown in table 2, for $h/\lambda = 0.1, 0.25$ and ∞, together with the corresponding values for $\theta = 0$. (If air compressibility is taken into account, it can easily be found, for fixed h/λ and θ, that the resonance value of a/λ increases with ξ.) The figures in table 2 show how the introduction of phase difference can drastically reduce the size of the system (the turbine size being doubly affected by the reductions of a/λ and C), at a (probably) moderate cost in terms of overall efficiency (theoretically at no cost for an isentropic process).

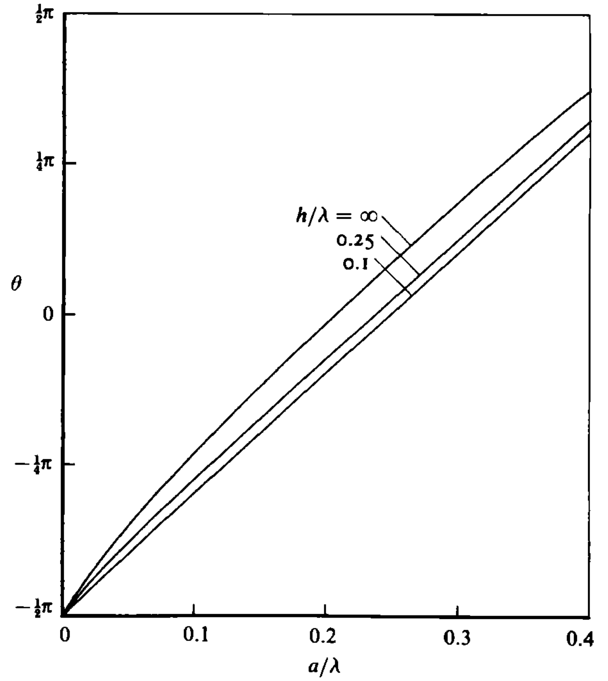

FIGURE 5. Resonance value of turbine phase difference θ as a function of dimensionless chamber length a/λ, for incompressible air and several values of dimensionless depth h/λ ((30a)).

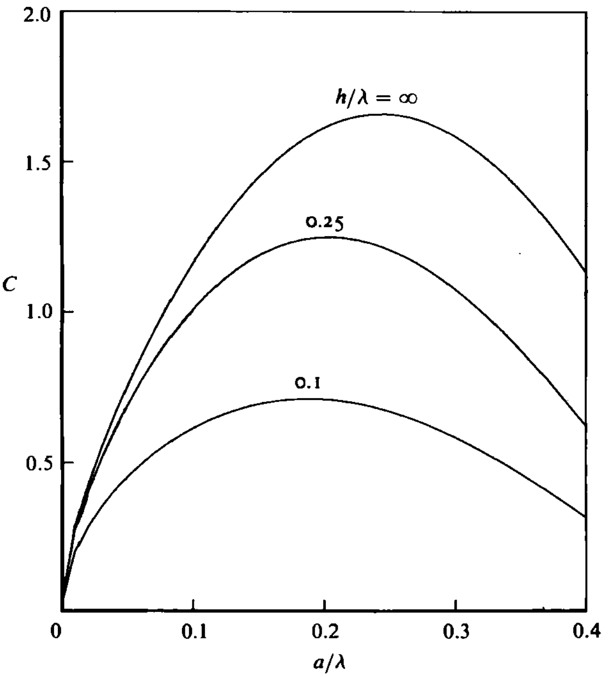

FIGURE 6. Resonance value of turbine coefficient C as a function of dimensionless chamber length a/λ, for incompressible air and several values of dimensionless depth h/λ (35).

	$\tau = 0.15$, $\theta = -0.999$ rad			$\theta = 0$		
h/λ	0.1	0.25	∞	0.1	0.25	∞
a/λ	0.090	0.079	0.064	0.249	0.237	0.207
C	0.59	0.88	0.88	0.67	1.22	1.63

TABLE 2. Resonance values of a/λ and C, for incompressible air and three water depths, in the cases of $\theta = -0.999$ rad ($\tau = 0.15$) and $\theta = 0$

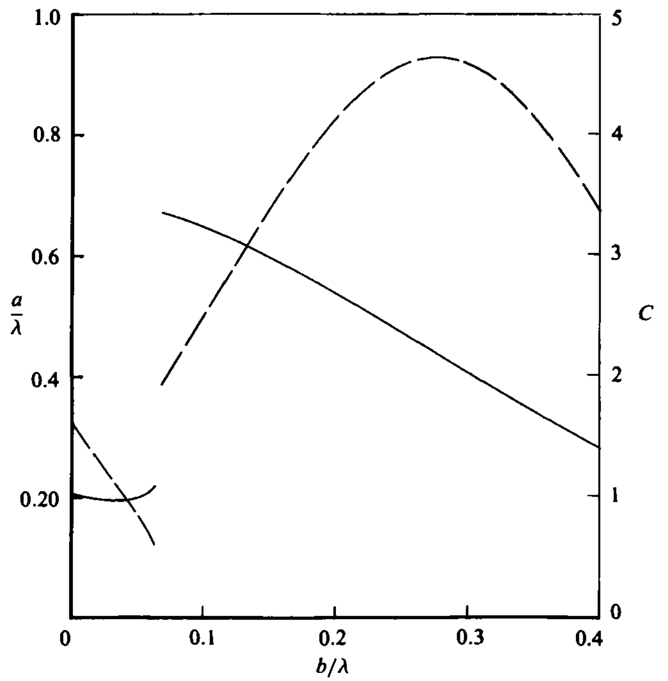

FIGURE 7. Resonance values of dimensionless chamber length a/λ ($\tau = 0$, solid line) and of turbine coefficient C ($\theta = 0$) ((34*b*), dashed line) plotted against the dimensionless distance b/λ between chamber and reflecting wall, for deep water and incompressible air.

4.2.2. *Chamber with non-adjacent wall*

Displacing the reflecting wall away from the chamber by a distance b not exceeding about 0.06λ can reduce substantially the resonance value of the turbine constant C, with little effect on the optimum chamber length a, as shown in figure 7, for deep water, incompressible air and $\theta = 0$.

5. Nonlinear problem

As mentioned in §3.1, non-zero terms of order higher than one are in general expected to appear in the Fourier expansion (15) of $P(t)$. As before, we consider the mass-flow rate M to be a known function Ψ of the pressure difference $P - P_a$. If the incident wave is sinusoidal, as we have been assuming, both sides of (14) are periodic functions of time, with period $2\pi/\omega$, and hence may be expanded in Fourier series. Equating the corresponding Fourier coefficients is equivalent to writing

$$\int_0^{2\pi/\omega} [\Psi(P-P_a)+i2aj\omega(H-z)\rho_c]\,e^{-ij\omega t}\,dt = 0, \quad j = 0,1,2,\ldots. \tag{36}$$

(If the chamber length is a the factor $2a$ is to be replaced by a.) Restricting the Fourier expansion (15) of $P(t)$ to terms of order up to $n = N$, we may consider P, z and ρ_c as known functions of t and of the vector $\boldsymbol{P}$ whose components are P_0 (real) and P_1, $P_2, \ldots, P_N$ (complex). (The expression of z is given by (16) or (22), and ρ_c is assumed related to P by the isentropic equation.) Consequently, (36) may be written in the form $F_j(\boldsymbol{P}) = 0$, where only equations up to $j = N$ are kept. The values of the coefficients P_n can, in principle, be determined by solving this system of nonlinear equations.

We saw that the efficiency ϵ can be expressed in terms of the ratios P_n/A $(n = 1,2,\ldots)$ (cf. §3). In the fully linear case, the amplitude of the pressure fluctuation is proportional to A, for a given turbine characteristic, and so the efficiency is independent of A, as seen in §4. This is not true for the nonlinear case, since (36) cannot be written in terms of P_n/A, except if the effect of air compressibility is linearized as in §4, and the function expressing the nonlinear turbine characteristic is adequately redefined in order to incorporate the influence of the wave amplitude. If this is done, the equations can be expressed in terms of the ratio $(P-P_a)/A$ (rather than P and A separately), and the efficiency becomes independent of A, as shown next.

We chose the case of a chamber of length a, adjacent to a reflecting wall, and assume the variation of ρ_c to be small and z to be negligible compared with H. Then, if ρ_c is linearized with respect to the pressure, as in §4, and z is replaced by its expression (22) (with $b = 0$), we easily find the following equations, which replace (36):

$$\left.\begin{aligned} &\int_0^{2\pi} G(\tau;\boldsymbol{\Pi})\,d\tau = 0,\\ &\int_0^{2\pi} G(\tau;\boldsymbol{\Pi})\,e^{-i\tau}\,d\tau + iB_1\Pi_1 - \frac{2i\pi}{K}\left(\frac{\alpha}{a}\right)^{\frac{1}{2}}\sin Ka = 0,\\ &\int_0^{2\pi} G(\tau;\boldsymbol{\Pi})\,e^{-ij\tau}\,d\tau + iB_j\Pi_j = 0, \quad j = 2,3,\ldots, \end{aligned}\right\} \tag{37}$$

where

$$B_j = j\pi(\alpha a)^{\frac{1}{2}}\left[\frac{\xi}{Ka} - \frac{1}{K_j a}(u_j + 2im_j \sin^2 K_j a)\right], \quad j = 1,2,3,\ldots, \tag{38}$$

ξ is given by (27), and $\boldsymbol{\Pi} = \{\Pi_0, \Pi_1, \Pi_2, \ldots\}$ consists of the Fourier coefficients of $\Pi = (P-P_a)/\rho g A$, i.e. $\Pi_0 = (P_0-P_a)/\rho g A$, $\Pi_j = P_j/\rho g A$ $(j = 1,2,\ldots)$. The dimensionless function G is defined by

$$\frac{M}{\rho_a A(ga)^{\frac{1}{2}}} = \frac{1}{\rho_a A(ga)^{\frac{1}{2}}}\Psi(P-P_a) = G(\tau;\boldsymbol{\Pi}), \tag{39}$$

where $\tau = \omega t$. In some cases in practice, Ψ may be considered an odd function; if so, it can be found from (37) that $\Pi_0 = \Pi_2 = \Pi_4 = \ldots = 0$.

Equation (38) shows that B_j is $O(j)$ or $O(j^{-1})$ according to whether air compressibility is taken into account or not (i.e. $\xi \neq 0$ or $\xi = 0$). Hence, from (37), we may conclude that air compressibility has the effect of reducing the order of magnitude of the higher harmonics and improving the convergence of the Fourier series. If a self-rectifying turbine (like the Wells turbine) is used, the pressure is expected to be a continuous function of time. Hence $\pi_j = O(j^{-n})$ as $j \to \infty$, and the contribution of the jth harmonic to the efficiency is $O(j^{-2n-1})$, where $n \geqslant 2$.

Numerical calculations were performed with the help of Brown's (1973) method for solving systems of nonlinear equations. We mention first that computations were made for several cases, in which air compressibility is important, by using the exact nonlinear isentropic relation between density and pressure, and also the linearized approximation described above. The results for the efficiency, obtained by both methods, were plotted together and found to be practically indistinguishable, except possibly for very large wave amplitudes, when anyway the linear surface-wave theory underlying the whole analysis is likely to provide only a poor approximation.

We consider next the case in which the mass-flow rate is proportional to the square root of the pressure difference. This happens approximately, for example, when an orifice is used to simulate the turbine in tests with small models. We write

$$M = \operatorname{sgn}(P - P_a)\, S\rho_a a(|P - P_a|/\rho)^{\frac{1}{2}},$$

where S is a dimensionless proportionality constant. From (39) it follows that $G(\tau; \Pi) = \operatorname{sgn}(\Pi)\Gamma|\Pi|^{\frac{1}{2}}$, where $\Gamma = S(a/A)^{\frac{1}{2}}$. (Here we assume A to be real and positive.) Results were computed for the case of a chamber of length a, in deep water, with an adjacent reflecting wall ($b = 0$), air-compressibility effects having been neglected ($\xi = 0$). Since Ψ is an odd function, $\Pi_j = 0$ for $j = 0, 2, 4, \ldots$, as pointed out above. Terms up to Π_9 were kept in the series, the computed values of $|\Pi_9|$ being less than $10^{-2}|\Pi_1|$. The calculations revealed that, if the truncation is made at $j = 5$, rather than $j = 9$, the changes in ϵ are $O(10^{-3})$, which indicates that even in this case the accuracy is satisfactory for engineering purposes. It was found by numerical iteration that, in the range $0 < a/\lambda < 0.5$, the maximum attainable efficiency is close to unity and occurs for $\Gamma \approx 0.118 = \Gamma_0$ (say) and a ratio a/λ which does not differ significantly from the resonance value 0.2066 for a chamber with a linear turbine (cf. table 1). The efficiency ϵ is plotted in figure 8 versus a/λ for $\Gamma = \Gamma_0$, $2\Gamma_0$, and $\frac{1}{2}\Gamma_0$. The first curve is to be compared with the curve, also shown in the figure, representing a chamber with an optimum linear turbine ($C = 1.628$, cf. table 1). The comparison reveals that the turbine nonlinearity here causes no significant drop in maximum efficiency. We note, however, that, in contrast with the linear case, the efficiency depends on the amplitude A of the incident wave except if Γ is kept constant, i.e. if the turbine coefficient S is made to vary proportionally to $A^{\frac{1}{2}}$.

If a conventional, rather than a self-rectifying, turbine is used, then the air flow has to be rectified by a system of valves, so that the flow through the turbine is unidirectional. In this case the turbine characteristic and also the timing of valve operation have to be specified. If the time taken to open and close the valves is much shorter than the wave period then the air pressure and flow rate may approximately be assumed as discontinuous functions of time.

Analytical results can easily be derived for the strongly nonlinear problem, considered next, in which the pressure is kept constant between jumps. This is approximately the case when the air circulates in closed circuit, with the water inside the chamber pumping air from a low-pressure reservoir to a high-pressure reservoir, between which an air turbine operates. We take the incident wave to be sinusoidal and defined by (13), and write $P - P_a = \pm \Pi A\rho g$, where Π is positive and independent of time. The jumps occur at $t = \theta + j\pi/\omega$ (j an integer) and are positive or negative respectively for j odd or even. We easily obtain $P_0 = P_a$, $P_n = 0$ for $n = 2, 4, 6, \ldots$, and

$$P_n = \mathrm{i}4\Pi A\rho g \frac{\mathrm{e}^{-\mathrm{i}n\theta}}{n\pi} \quad \text{for } n \text{ odd.}$$

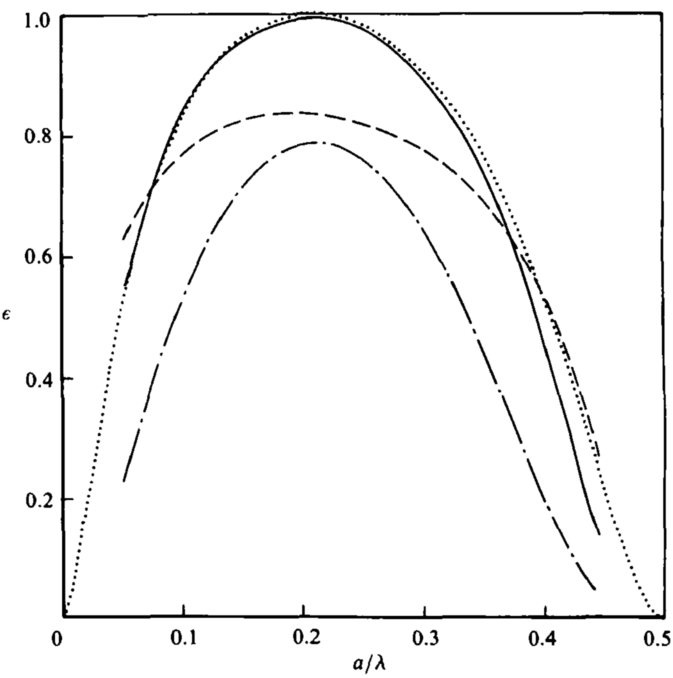

FIGURE 8. Variation of efficiency ϵ with dimensionless chamber length a/λ, in the case of nonlinear turbine characteristic of square-root type, for deep water and incompressible air: ——, $\Gamma = 0.118 = \Gamma_0$; ----, $0.5\Gamma_0$; —·—, $2\Gamma_0$. The dotted line corresponds to a linear turbine with $C = 1.628$, $\theta = 0$.

We further assume that the chamber is located at $-a < x < 0$, adjacent to a vertical reflecting wall at $x = 0$. If use is made of the expressions derived in §3 for the efficiency ϵ, we find

$$\epsilon = 2\Pi B_1 \cos\theta - \Pi^2 B^2, \tag{40}$$

where

$$B_n = \frac{8}{\pi}\left(\frac{m m_n K}{n K_n}\right)^{\frac{1}{2}} \sin K_n a, \quad B^2 = \sum_{n=1}^{\infty} B_{2n-1}^2.$$

It can be found that $B_n^2 = O(n^{-3})$, which means that, even in this case, a good accuracy may be expected by keeping only a few harmonics. Assuming $B_1 > 0$, i.e. $a/\lambda < \frac{1}{2}$, the value θ that yields the highest efficiency is obviously $\theta = 0$. This means that the opening of the valves connecting the chamber to the low-pressure reservoir or the high-pressure reservoir should coincide in time respectively with the highest or the lowest elevation, inside the chamber, of the standing wave resulting from the superposition of the incident wave and the wave reflected on the wall (i.e. not considering the applied-pressure radiation wave). Setting $\theta = 0$ in (40), we find that, for fixed a/λ and h/λ, the value of Π maximizing the efficiency is $\Pi = B_1/B^2$, which gives $\epsilon = (B_1/B)^2$. These values of ϵ and Π are plotted in figure 9 as functions of a/λ, for deep water. The efficiency has its maximum value, equal to 0.997, for $a/\lambda = 0.222$ and 0.278, the corresponding value of Π being 0.398.

6. Conclusions

An analysis, based on linear surface-wave theory, has been presented to describe the performance of an oscillating-water-column device of simple two-dimensional geometry, assuming the incident wave to be sinusoidal. Special attention is dedicated

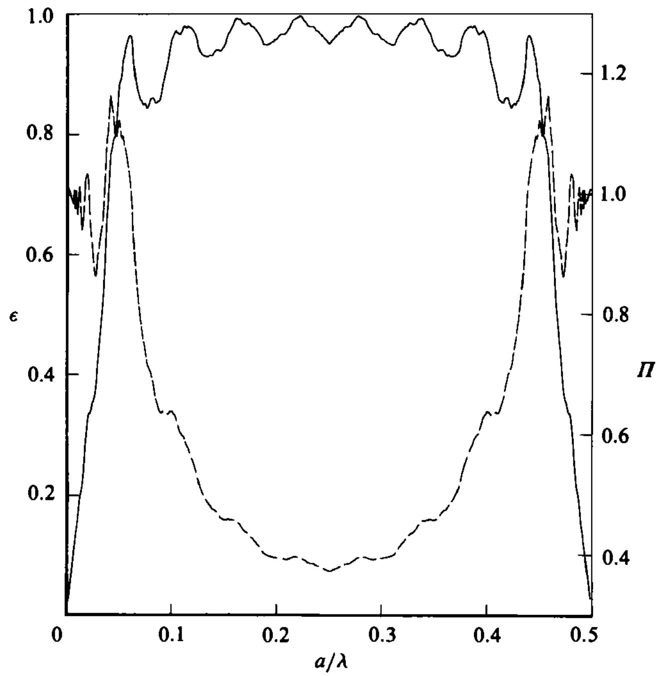

FIGURE 9. Maximum efficiency ϵ (solid line) and corresponding dimensionless pressure Π (broken line) shown as functions of dimensionless chamber length a/λ, in the case of discontinuous time variation of pressure of rectangular-wave type.

to the fully linear case. It is shown that air compressibility can affect significantly the performance of full-scale devices whose chamber 'height' H (air volume divided by inside free-surface area) is expected to attain several metres. If this effect is to be adequately represented when testing small models, H (more precisely $\rho g H/\gamma P_a$) must be equal to the corresponding full-scale value, a point that has been disregarded in the past. Linearizing the springlike air-compressibility effect can provide a satisfactory approximation to what is obtained by using the nonlinear isentropic pressure–density relation. We point out that, anyway, the compression and expansion of air are more complex and likely to deviate appreciably from an isentropic process, owing to viscous losses in the flow through the turbine.

Theoretically, providing the turbine with phase control enables the same (maximum) amount of energy to be extracted from the waves regardless of how short the chamber may be. It is shown that the chamber size (and also the turbine size) can be reduced very substantially, before the drop in overall efficiency of the real turbine becomes unacceptable as a consequence of the joint effect of viscous losses and increasing phase difference. Efficiency equal to unity can be attained theoretically if a reflecting wall is present and provided that the turbine has a linear characteristic. However, it is found that efficiencies close to unity can still be achieved even in cases of strongly nonlinear power take-off. If a nonlinear characteristic is given, the efficiency is dependent on the amplitude of the incident wave (unlike the fully linear case), this implying that a nonlinear power take-off system has to be tuned to the wavelength *and* to the wave amplitude.

The present analysis can be extended in order to include the more general case when the incident wave is time-periodic, by simply adding the higher harmonics to the incident wave field. This can provide a better approximation to the performance of a device in irregular real sea waves. We note that, if the turbine is capable of phase control, this means that the pressure can be controlled independently of the flow rate within some finite range. If this range is wide enough and a reflecting wall is present to prevent energy being lost as transmitted waves, then it seems theoretically possible to extract the whole energy from an irregular incident wave by making the radiated wave cancel out the wave reflected on the wall. This appears to be a promising direction of study, to which the present work can provide some ground.

In a different way, the present theory can be extended to apply to more complex two-dimensional geometries if results for the diffraction and applied-pressure radiation fields are known. For example, the effect of having the chamber lips submerged to a finite depth can be accounted for by using the results of Ogilvie (1969). We point out that only the radiation potential needs to be given, since the relation of Haskind type derived by Fernandes (1983, equation (2.16)) makes it possible to calculate the air flux due to the diffraction wave field. If, instead, the diffraction, rather than the radiation, problem is solved, the relation derived by Evans (1982, equation (2.28)) allows the real part of the complex admittance to be determined, but not the imaginary part, which is also required if the device performance is to be calculated for a given power take-off characteristic. The imaginary part can possibly be obtained by relations of Kramers–Krönig type, but this is inadequate for computational purposes, as pointed out by Fernandes (1983).

Finally, we note that, although a considerable degree of simplification has been introduced in the assumptions that underlie the present analysis, particularly the choice of geometry, it seems reasonable to expect that some results derived here will apply approximately, or at least qualitatively, to more complex situations and be relevant to certain wave-energy devices which are presently being considered in several countries.

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